UDC 539.3

ON A DUAL VARIATIONAL PRINCIPLE IN GEOMETRICALLY NONLINEAR ELASTICITY THEORY

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A method for constructing a dual variational principle in geometrically nonlinear elasticity theory is elucidated. The dual functional (a functional of Castigliano type) is evaluated in the case of an isotropic semilinear material.

1. Introduction. The true displacement field in geometrically linear elasticity theory is the extremal of a Lagrange functional $I(u)$ defined on admissible displacement fields u

$$\delta I(u) = 0 \tag{1.1}$$

and the true stress field is the extremal of the Castigliano functional $J(\sigma)$, defined on the admissible stress fields σ

$$\delta J(\sigma) = 0 \tag{1.2}$$

It is essential that the Lagrange and Castigliano functionals form a pair of dual functionals. This means that in addition to (1.1) and (1.2), the equation

$$\inf_u I(u) = \sup_\sigma J(\sigma) \tag{1.3}$$

is valid. The equality (1.3) permits the construction of two-sides estimates of the elastic energy [1]. Two-sided estimates of the effective elastic moduli of micro-inhomogeneous elastic bodies or inequalities permitting the proof of the asymptotic accuracy of applied theories of rods, plates, and shells result from these estimates, for example.

Because of the nonconvexity, it is difficult to depend on the existence of a functional simultaneously satisfying the conditions (1.2) and (1.3) in the geometrically nonlinear theory of elasticity. Zubov [2] constructed a functional $J(\sigma)$ satisfying condition (1.2). Koiter [3] turned attention to the fact that the Legendre transformation used in [2] results in a multivalued functional and it remains unclear how the branch of $J(\sigma)$ should be selected. The Zubov paper [4] is devoted to this question.

A number of questions associated with the construction of the functional $J(\sigma)$ was discussed in [5 —13]. Let us also mention the works on variational problems with nonconvex functionals [14, 15].

A functional $J(\sigma)$ is constructed below, which does not generally satisfy (1.2), however, is subject to the relationship

$$\sup_\sigma J(\sigma) \leqslant \inf_u I(u) \tag{1.4}$$

An important property of the dual. variational principle is hence conserved, the possibility of constructing bilaterial energy estimates.

2. Lagrange functional. Let us consider a geometrically nonlinear

body occupying a domain V_0 with piecewise-smooth boundary ∂V_0 in the undeformed state. Let x^i denote the coordinates of the Cartesian reference system of the observer, and ξ^p the coordinates of an accompanying reference system. The indices by which manipulations occur during manipulation of the observer reference system are marked by the symbols $i, j, k, \ldots$, while manipulations in the accompanying coordinate system are marked by $p, q, r, \ldots$ The locations of the particles in the undeformed and deformed states are given by the functions $x_0{}^i = x_0{}^i(\xi^p)$ and $x^i = x^i(\xi^p)$, respectively. The boundary ∂V_0 is separated into two parts, S and Σ, where surface "dead" loads p_i are given on S and the positions of the particles in the deformed state

$$x^i(\xi^p)|_\Sigma = \varphi^i(\xi^p) \tag{2.1}$$

are given on Σ.

Let us define the functional [16]

$$I[x^i(\xi^p)] = E(x^i) - l(x^i) \tag{2.2}$$

$$E(x^i) = \int_{V_0} U(x_p{}^i)\,d\tau, \quad l(x^i) = \int_{V_0} F_i x^i(\xi^p)\,d\tau + \int_{\partial V_0} p_i x^i(\xi^p)\,d\sigma$$

Here U is the internal energy density, $x_p{}^i = \partial x^i / \partial \xi^p$ is the strain gradient, F_i is the mass force density. Later we will assume that the minimum value of the functional (2.2) is achieved on functions satisfying the condition

$$\det \| x_p{}^i \| > 0 \tag{2.3}$$

The equilibrium positions of bodies are critical points of the functional (2.2) in a set of functions satisfying conditions (2.1).

Usually U is given in the geometrically nonlinear theory of elasticity as the convex function ε_{pq} or γ_{pq}

$$\varepsilon_{pq} = {}^1/_2\,(g_{ij}x_p{}^i x_q{}^j - g_{pq}{}^\circ), \quad \gamma_{pq} = |\,x\,|_{pq} - |\,x_0\,|_{pq} \tag{2.4}$$

$$g_{pq}{}^\circ = g_{ij}x_{0p}{}^i x_{0q}{}^j, \quad x_{0p}{}^i = \partial x_0{}^i / \partial \xi^p, \quad x_p{}^i = |\,x\,|_{pq}\lambda^{iq}$$

Here g_{ij} and $g_{pq}{}^\circ$ are convariant components of the metric tensor in the observer reference system and in the accompanying reference system in the undeformed state, $|\,x\,|_{pq}$ is the modulus of the tensor $x_p{}^i$ defined by the Cayley polar expansion [17], $|\,x\,|_{pq}$ is a positive-definite symmetric tensor, λ^{iq} are components of the orthogonal matrix

$$g_{pq}{}^\circ \lambda^{ip}\lambda^{jq} = g^{ij}, \quad g_{ij}\lambda^{ip}\lambda^{jq} = g_0{}^{pq} \tag{2.5}$$

$|\,x_0\,|_{pq}$ is the modulus of the tensor $x_{0p}{}^i$. Juggling of the indices $p, q, r, \ldots$ is accomplished by using the metric $g_{pq}{}^\circ$, and the indices $i, j, k, \ldots$ by using the metric g_{ij}.

If the function U were convex in $x_p{}^i$, then the dual variational problem could be constructed according to the general rule formulated in [18]. However, U while convex in ε_{pq} or γ_{pq} is not convex in $x_p{}^i$.

The following examples show this.

Example 1. Let us consider one-dimensional strain: $x^1 = x^1(\xi^1)$, $x^2 = \xi^2$, $x^3 = \xi^3$, $g_{11}{}^0 = 1$ (we later omit the superscript 1). Let U be a quadratic form in

ε_{pq}. Then $U = E\varepsilon^2$, $\varepsilon = \varepsilon_{11} = {}^1/_2\,[(\partial x / \partial \xi)^2 - 1]$ in the case under consideration. The nonconvexity of U in $\partial x / \partial \xi$ is evident.

Example 2. Let us consider an isotropic elastic semilinear material for which the internal energy density U has the form

$$U(\gamma_{pq}) = {}^1/_2\lambda\,(g_0^{pq}\gamma_{pq})^2 + \mu\,(\gamma^{pq}\gamma_{pq})$$

Taking (2.4) into account for the two-dimensional problem, we write the function U as a function of x_p^α, $(\alpha, p = 1, 2)$ for $\lambda = 0$ in the form

$$U(x_p^\alpha) = \mu\,\{(x_1^1)^2 + (x_2^1)^2 + (x_1^2)^2 + (x_2^2)^2 - 2\,[(x_1^1 + x_2^2)^2 + (x_2^1 - x_1^2)^2]^{1/2} + 2\}$$

A necessary condition for convexity of a function of many variables is its convexity in each of the variables. Verifying this condition for the function (2.6) reduces to investigating the sign of the second partial derivative of U with respect to each of the variables x_p^α. For instance, let us write down $\partial^2 U / \partial (x_1^1)^2$

$$\frac{1}{2\mu}\frac{\partial^2 U}{\partial (x_1^1)^2} = 1 - \frac{(x_2^1 - x_1^2)^2}{[(x_1^1 + x_2^2)^2 + (x_2^1 - x_1^2)^2]^{3/2}}$$

It is clear that x_2^1, x_1^2 and x_2^2 can be selected in such a manner that $\partial^2 U / \partial (x_1^1)^2 < 0$. This indicates the nonconvexity of the function U in x_p^i.

The static instability of elastic bodies as well as the presence of several equilibrium positions are related to the nonconvexity of the function U with respect to x_p^i.

3. Dual variational principle. Let us consider the problem of the minimum of the functional (2.2) in all the functions satisfying the conditions (2.1). Let us assume that the minimum value of the variational problem (2.2), (2.1) is achieved by functions satisfying conditions (2.3).

Let U be an arbitrary continuously-differentiable function of γ_{pq} (or equivalently, of $|x|_{pq}$). Let us consider the function

$$\bar{U} = \begin{cases} U(x_p^i), & x_p^i:\ \det\|x_p^i\| > 0 \\ +\infty, & x_p^i:\ \det\|x_p^i\| \leqslant 0 \end{cases} \tag{3.1}$$

Let $\bar{U}^*(\sigma_i^p)$ be the Young transformation of the functions $\bar{U}(x_p^i)$ relative to x_p^i

$$\bar{U}^*(\sigma_i^p) = \sup_{x_p^i}\,[\sigma_i^p x_p^i - \bar{U}(x_p^i)] \tag{3.2}$$

Here σ_i^p are dual variables. Let $\bar{U}^{**}(x_p^i)$ be the Young transformation of the functions $\bar{U}^*(\sigma_i^p)$

$$\bar{U}^{**}(x_p^i) = \sup_{\sigma_i^p}\,[x_p^i\sigma_i^p - \bar{U}^*(\sigma_i^p)] \tag{3.3}$$

It is known [19] that $\bar{U}^{**}(x_p^i)$ is the maximum convex function in x_p^i that does not exceed $\bar{U}(x_p^i)$.

Let us introduce the notation

$$I(x^i) = \int_{V_0} \bar{U}(x_p^i)\,d\tau - l(x^i), \quad E^*(\sigma_i^p) = \int_{V_0} \bar{U}^*(\sigma_i^p)\,d\tau$$

$$l^*(\sigma_i^p) = \int_{V_0} \sigma_i^p \frac{\partial x_\Sigma^i}{\partial \xi^p}\,d\tau - l(x_\Sigma^i)$$

where $x_\Sigma{}^i$ are arbitrary functions satisfying condition (2.1).

We formulate the dual variational principle as a problem on the minimum of the functional

$$J(\sigma_i{}^p) = E^*(\sigma_i{}^p) - l^*(\sigma_i{}^p)$$

where the minimum is considered in all the $\sigma_i{}^p$ satisfying the conditions

$$\int_{V_0} \sigma_i{}^p \frac{\partial x'^i}{\partial \xi^p} d\tau - l(x'^i) = 0 \qquad (3.4)$$

By the condition, (3.4) holds for any functions x'^i which vanishes on Σ. If $\sigma_i{}^p$ are continuously differentiable, then the constraints (3.4) can be rewritten in the form (n_p is the normal to S)

$$\partial\sigma_i{}^p / \partial\xi^p + F_i = 0 \quad \text{in } V_0, \qquad \sigma_i{}^p n_p = p_i \quad \text{on } S \qquad (3.5)$$

Let us note that values of the functional $l^*(\sigma_i{}^p)$ are independent of the selection of the functions $x_\Sigma{}^i$ because of (3.4).

T h e o r e m 1. The following inequality is valid

$$\sup_{\sigma_i{}^p \in (3.5)} [-J(\sigma_i{}^p)] \leqslant \inf_{x^i \in (2.1)} I(x^i)$$

Writing $x^i \in (2.1)$ means that x^i satisfies the conditions (2.1).

P r o o f. From the assumption that the minimizing element of the variational problem (2.1), (2.2) satisfies condition (2.3), we have

$$\inf_{x^i \in (2.1)} I(x^i) = \inf_{x^i \in (2.1)} \bar{I}(x^i) \geqslant \inf_{x^i \in (2.1)} \Big[\int_{V_0} \bar{U}^{**} d\tau - l(x^i) \Big] =$$

$$\inf_{x^i (2.1)} \sup_{\sigma_i{}^p} \Big[\int_{V_0} (\sigma_i{}^p x_p{}^i - \bar{U}^*) d\tau - l(x^i) \Big] \geqslant$$

$$\sup_{\sigma_i{}^p \in (3.5)} \inf_{x^i \in (2.1)} \Big[\int_{V_0} \sigma_i{}^p x_p{}^i d\tau - l(x^i) - \int_{V_0} \bar{U}^* d\tau \Big] =$$

$$\sup_{\sigma_i{}^p \in (3.5)} [l^*(\sigma_i{}^p) - E^*(\sigma_i{}^p)] = \sup_{\sigma_i{}^p \in (3.5)} [-J(\sigma_i{}^p)]$$

Here we used a valid inequality for any functional $\Phi(x, \sigma)$

$$\inf_x \sup_\sigma \Phi(x, \sigma) \geqslant \sup_\sigma \inf_x \Phi(x, \sigma)$$

Therefore, the problem of constructing a dual variational principle reduces to calculating the Young transformation of the function $\bar{U}$.

4. O n t h e Y o u n g t r a n s f o r m a t i o n o f t h e f u n c t i o n $\bar{U}$. Let us consider the Young transformation of the function $\bar{U}$ in $x_p{}^i$ (3.2). Let us note that

$$\bar{U}^*(\sigma_i{}^p) = \sup_{x_p{}^i \in (2.3)} [\sigma_i{}^p x_p{}^i - U(x_p{}^i)] \qquad (4.1)$$

L e m m a 1. Let θ^{pq} be an arbitrary tensor, and μ_{pq} a tensor satisfying the orthogonality conditions

$$g_0{}^{pq} \mu_{ps} \mu_{qt} = g_{st}{}^\circ, \quad \det \| \mu_{pq} \| > 0 \qquad (4.2)$$

Then

$$\sup_{\mu_{pq}\in(4.2)} \theta^{pq}\mu_{pq} = |\theta|_1 + |\theta|_2 + |\theta|_3 \operatorname{sgn} \det \| \theta^{pq}\|$$

Here $|\theta|_i$ are the eigenvalues of the tensor $|\theta|^{pq}$ arranged in decreasing order.

P r o o f. 1°. First let $\det \|\theta^{pq}\| > 0$. Using the polar expansion of the tensor θ^{pq}, i.e., $\theta^{pq} = |\theta|^{ps}\lambda_s{}^q$, the assertion of the lemma can be rewitten in the form

$$\sup_{\nu_{st}\in(4.2)} |\theta|^{ps} \nu_{ps} = |\theta|_1 + |\theta|_2 + |\theta|_3 \tag{4.3}$$

The tensor ν_{st} which satisfies (4.2) in a smooth coordinate system for the symmetric tensor $|\theta|^{ps}$ is represented in the form of the product of three elementary rotations with independent Euler parameters φ_1, φ_2, ψ.

The proof of (4.3) reduces to evaluating the quantities

$$\theta = \sup_{\varphi_1,\varphi_2,\psi} [A \cos\varphi_1 \cos\varphi_2 - B \sin\varphi_1 \sin\varphi_2 + |\theta|_3 \cos\psi] \tag{4.4}$$

$$A = |\theta|_1 + |\theta|_2 \cos\psi, \quad B = |\theta|_1 \cos\psi + |\theta|_2$$

Evaluation of the upper bound in φ_2 yields

$$\theta = \sup_{\varphi_1\psi} [(A^2 \cos^2\varphi_1 + B^2 \sin^2\varphi_1)^{1/2} + |\theta|_3 \cos\psi]$$

Since $A \geqslant 0$, $A^2 \geqslant B^2$ (let us recall that $|\theta|_1 \geqslant |\theta|_2 \geqslant |\theta|_3$), A and B are independent of φ_1, we obtain

$$\theta = \sup_\psi (A + |\theta|_3 \cos\psi) = |\theta|_1 + |\theta|_2 + |\theta|_3$$

2°. Let $\det \|\theta^{pq}\| < 0$. In this case the assertion of the lemma has the form

$$\sup_{\varkappa_{st}} |\theta|^{st} \varkappa_{st} = |\theta|_1 + |\theta|_2 - |\theta|_3$$

where the upper bound is considered in all the $\varkappa_{st}$ satisfying the conditions

$$g_0{}^{pq} \varkappa_{ps}\varkappa_{qt} = g_{st}{}^\circ, \quad \det \|\varkappa_{pq}\| < 0 \tag{4.5}$$

In this case the proof of the assertion is analogous to the proof in case 1°. It is just necessary to reverse the sign before the term $|\theta|_3 \cos\psi$ in (4.4).

R e m a r k. Let us note that the assertion of Lemma 1 is not related to the ordering of the eigenvalues for $\det \|\theta^{pq}\| > 0$. When $\det \|\theta^{pq}\| < 0$, the assertion of the lemma can also be represented in a form not assuming the ordering of the eigenvalues

$$\sup_{\mu_{pq}\in(4.2)} \theta^{pq}\mu_{pq} = \max [|\theta|_1 + |\theta|_2 - |\theta|_3, |\theta|_2 + |\theta|_3 - |\theta|_1, |\theta|_3 + |\theta|_1 - |\theta|_2]$$

L e m m a 2. Let U depend only on the eigenvalues of the tensor $|x|_{pq}$ and let $|\sigma|^{pq}$ be the modulus of the tensor $\sigma_i{}^p$. Then the tensors $|x|_{pq}$ and $|\sigma|^{pq}$ are coaxial.

T h e o r e m 2. Let the function U satisfy the conditions of Lemma 2. Then the Young's transformation of the function $\bar U$ is given in a smooth coordinate system for the tensor $|\sigma|^{pq}$, by the formula

$$\bar U^* (\sigma_i{}^p) = \begin{cases} U_1{}^*, & \det \|\sigma_i{}^p\| > 0 \\ U_2{}^*, & \det \|\sigma_i{}^p\| \leqslant 0 \end{cases} \tag{4.6}$$

$$U_1{}^* = \sup_{|x|_i>0} [|x|_i |\sigma|^i - U(|x|_i)]$$

$$U_2{}^* = \sup_{|x|_i>0} [\max(\psi_1, \psi_2, \psi_3) - U(|x|_i)]$$

$$\psi_1 = |x|_3 |\sigma|^3 + |x|_2 |\sigma|^2 - |x|_1 |\sigma|^1$$
$$\psi_2 = |x|_3 |\sigma|^3 + |\sigma|_1 |x|^1 - |x|_2 |\sigma|^2$$
$$\psi_3 = |x|_1 |\sigma|^1 + |x|_2 |\sigma|^2 - |x|_3 |\sigma|^3$$

where $|x|_i$ and $|\sigma|_i$ are, respectively, the eigenvalues of the tensors $|x|_{pq}$ and $|\sigma|^{ps}$.

P r o o f. Let us substitute the polar expansion of the tensors $x_p{}^i$ and $\sigma_i{}^p$ in (4.1)

$$\bar{U}^* (\sigma_i{}^p) = \sup_{x_p{}^i \in (2.3)} [|\sigma|^{pq} |x|_{ps} \nu_q{}^s - U(|x|_{pq})] \qquad (4.7)$$

where $\nu_q{}^s$ is a tensor satisfying the conditions

$$g_0{}^{pq} \nu_p{}^s \nu_q{}^t = g_0{}^{st}, \quad \operatorname{sgn} \det \| \nu_q{}^s \| = \operatorname{sgn} \det \| \sigma_i{}^p \| \qquad (4.8)$$

Let us evaluate the upper bound in (4.7) successively by first taking the upper bound in $\nu_q{}^s$ and then in $|x|_{pq}$. Since U is independent of $\nu_q{}^s$, then finding the upper bound in $\nu_q{}^s$ reduces to evaluating the quantity

$$\sup_{\nu_q{}^s \in (4.8)} \theta_s{}^p \nu_p{}^s, \quad \theta_s{}^p = |\sigma|^{pq} |x|_{qs} \qquad (4.9)$$

From the coaxiality of the tensors $|\sigma|^{pq}$ and $|x|_{rs}$ the eigenvalues of the tensor $|\theta|^{pq}$ have the form

$$|\theta|^i = |x|^i |\sigma|^i, \quad i = 1, 2, 3 \qquad (4.10)$$

whereupon (4.9), (4.10) and Lemma 1 result in the assertion of the theorem.

In the case of an anisotropic elastic medium, the evaluation of $\bar{U}^*$ is a complex problem. Let us mention a simple method of estimating the minimum value of the functional (2.3) in this case.

Let $U(|x|_{pq})$ be the internal energy density, $s_1(|x|_{pq})$, $s_2(|x|_{pq})$, and $s_3(|x|_{pq})$ the first, second, third invariants, respectively, of the tensor $|x|_{pq}$. Let us consider the function

$$U_1(r_1, r_2, r_3) = \inf_{|x|_{pq} : s_i = r_i, \ i = 1, 2, 3} U(|x|_{pq})$$

Evidently $U_1 \leqslant U$. The function U_1 can be considered as the internal energy density of some iostropic elastic material whose elastic characteristics are "less" than the elastic characteristics of the initial anisotropic material.

Let us introduce the functional

$$I_1(x^i) = \int_{V_0} U_1 \, d\tau - l(x^i)$$

and let J_1 denote the functional dual to I_1. Then

$$\sup_{\sigma_i{}^p \in (3.8)} (-J_1) \leqslant \inf_{x^i \in (2.1)} I_1 \leqslant \inf_{x^i \in (2.1)} I$$

5. The function $\bar{U}^*$ for a semilinear material. 1°. The function $\bar{U}^*$ is evaluated especially simply in the case of a semilinear material with zero Lamé coefficient λ. We have from (4.6)

$$U_1{}^* = {}^1/_4 \mu^{-1} [|\sigma|_1{}^2 + |\sigma|_2{}^2 + |\sigma|_3{}^2] + |\sigma|_1 + |\sigma|_2 + |\sigma|_3$$

and $U_2{}^*$ is the maximum of the values at the point $|\sigma|_i$ for the function

$$\psi(|\sigma|_1, |\sigma|_2, |\sigma|_3),\ \psi(|\sigma|_3, |\sigma|_1, |\sigma|_2),\ \psi(|\sigma|_2, |\sigma|_3, |\sigma|_1),$$

where

$$\psi(\alpha_1, \alpha_2, \alpha_3) = {}^1/_4\,\mu^{-1}[\alpha_1^2 + \alpha_2^2 + \beta^2] + \alpha_1 + \alpha_2 - \beta$$

$$\beta = \begin{cases} \beta = \alpha_3, & \alpha_3 \leqslant 2\mu \\ \beta = 2\mu, & \alpha_3 > 2\mu \end{cases}$$

2°. Let us examine the plane case: $x^\alpha = x^\alpha(\xi^1, \xi^2)$, $x^3 = \xi^3$ (the Greek superscripts run through the values 1, 2). Then $|\sigma|_3 = 0$, $\gamma_3 = 0$, and the Young transformation of $U(|x|_\alpha)$ has the form

$$U_1^* = {}^1/_4\mu^{-1}[|\sigma|_1^2 + |\sigma|_2^2] + |\sigma|_1 + |\sigma|_2$$

$$U_2^* = \begin{cases} {}^1/_4\,\mu^{-1}[|\sigma|_1^2 + |\sigma|_2^2] + |\sigma|_1 - |\sigma|_2, & |\sigma|_\alpha \in \Sigma_{12} \\ {}^1/_4\,\mu^{-1}|\sigma|_1^2 + |\sigma|_1 - \mu, & |\sigma|_\alpha \in \Sigma_1 \end{cases} \qquad (1 \leftrightarrow 2)$$

where Σ_{12}, Σ_{21}, Σ_1, Σ_2 are domains in the plane $(|\sigma|_1, |\sigma|_2)$ indicated in Fig. 1, and $|\sigma|_\alpha$ are eigenvalues of the tensor $|\sigma|_{pq}$.

The function $\bar{U}^{**}(x_p^i)$ permits an assessment of the nature of the roughness occurring in going over to the dual variational problem. If $\bar{U}^{**} = \bar{U}$, then the dual variational problem is an accurate reformulation of the initial problem. If $\bar{U}^{**}$ differs negligibly from $\bar{U}$, then it can be expected that the solution of the dual problem will differ negligibly from the initial problem in the energy norm.

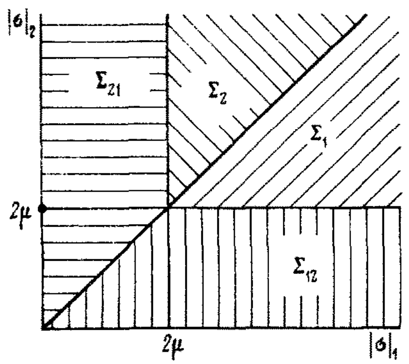

Fig. 1

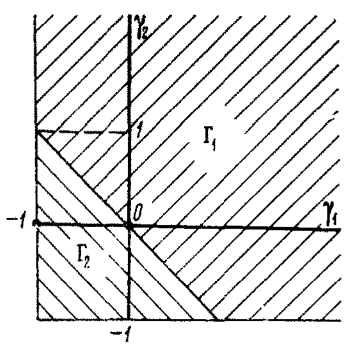

Fig. 2

A calculation by means of (3. 3) yields

$$\bar{U}^{**}(x_p^i) = \begin{cases} \mu(\gamma_1^2 + \gamma_2^2), & \gamma_\alpha \in \Gamma_1 \\ {}^1/_2\,\mu(\gamma_1 - \gamma_2)^2, & \gamma_\alpha \in \Gamma_2 \end{cases}$$

where γ_α are the eigenvalues of the tensor $\gamma_{\alpha\beta}$, Γ_1, Γ_2 are domains of the plane (γ_1, γ_2) indicated in Fig. 2.

Therefore, $\bar{U}^{**} = \bar{U}$ if the deformations lie in the domain Γ_1. This domain consists of tensions along both axes as well as compression along one axis and tension along the other for which the tension is greater than the compression. In the domain Γ_2 (all the remaining cases) $\bar{U}^{**} < \bar{U}$, where in Γ_2

$$\bar{U} - \bar{U}^{**} = {}^1/_2\mu\,(\gamma_1 + \gamma_2)^2 \qquad (5.1)$$

If a solid is subjected to deformations for which γ_α lie in the domain Γ_1, then the dual principle yields the same solution as the initial.

If the true deformations γ_1, γ_2 lie in the domain Γ_2, then the dual principle will not result in the exact solution. The nature of the energy error is determined by (5.1).

3° Let us present the result of evaluating the function $\bar{U}^*$ for $\lambda \neq 0$. According to Theorem 2

$$\bar{U}^*(\sigma_i^p) = \begin{cases} U_1^*(|\sigma|_1, |\sigma|_2, |\sigma|_3), & \sigma_i^p: \det\|\sigma_i^p\| \geqslant 0 \\ U_2^*(|\sigma|_1, |\sigma|_2, |\sigma|_3), & \sigma_i^p: \det\|\sigma_i^p\| < 0 \end{cases}$$

where $U_2^*(|\sigma|_i)$ is the maximum value at the point $|\sigma|_i$ for the functions $U_1^*(-|\sigma|_1, |\sigma|_2, |\sigma|_3)$. $U_1^*(|\sigma|_1, -|\sigma|_2, |\sigma|_3)$, $U_1^*(|\sigma|_1, |\sigma|_2, -|\sigma|_3)$.

Let us divide the domain of variation of the variables $|\sigma|_1, |\sigma|_2, |\sigma|_3$ into ten domains $\Sigma, \Sigma_1, \Sigma_1', \Sigma_1'', \Sigma_2, \Sigma_2', \Sigma_2'', \Sigma_3, \Sigma_3', \Sigma_3''$. The domains Σ, $\Sigma_1\Sigma_1'\Sigma_1''$ are defined by the relationships

$$\Sigma: |\sigma|_1 \geqslant \nu(|\sigma|_2 + |\sigma|_3) - E, \quad |\sigma|_2 \geqslant \nu(|\sigma|_1 + |\sigma|_3) - E, \quad |\sigma|_3 \geqslant \nu(|\sigma|_1 + |\sigma|_2) - E \qquad (5.2)$$

$$\Sigma_1: |\sigma|_1 < \nu(|\sigma|_2 + |\sigma|_3) - E, \quad |\sigma|_2 \geqslant \sigma|\sigma|_3 - E/(1-\nu), \quad |\sigma|_3 \geqslant \sigma|\sigma|_2 - E/(1-\nu)$$

$$\Sigma_1': |\sigma|_1 < \nu(|\sigma|_2 + |\sigma|_3) - E, \quad 0 \leqslant |\sigma|_2 < \sigma|\sigma|_3 - E/(1-\nu)$$

$$\Sigma_1'': |\sigma|_1 < \nu(|\sigma|_2 + |\sigma|_3) - E, \quad 0 \leqslant |\sigma|_3 < \sigma|\sigma|_2 - E/(1-\nu)$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \quad E = \frac{3\lambda + 2\mu}{\lambda+\mu}\mu, \quad \sigma = \frac{\lambda}{\lambda+2\mu}$$

In these domains the function $U_1^*(|\sigma|_i)$ has the form

$$^1/_4\mu^{-1}\left[|\sigma|_1^2 + |\sigma|_2^2 + |\sigma|_3^2 - \frac{\lambda}{3\lambda+2\mu}(|\sigma|_1 + |\sigma|_2 + |\sigma|_3)^2\right] + |\sigma|_1 + |\sigma|_2 + |\sigma|_3 \quad \text{in } \Sigma \qquad (5.3)$$

$$^1/_4\mu^{-1}\left[|\sigma|_2^2 + |\sigma|_3^2 - \frac{\lambda}{2(\lambda+\mu)}(|\sigma|_2 + |\sigma|_3 - 2\mu)^2\right] + |\sigma|_2 + |\sigma|_3 - \mu \quad \text{in } \Sigma_1$$

$$\frac{(|\sigma|_3 + 2\lambda)^2}{2(\lambda+2\mu)} + |\sigma|_3 - 2(\lambda+\mu) \quad \text{in } \Sigma_1'$$

$$\frac{(|\sigma|_2 + 2\lambda)^2}{2(\lambda+2\mu)} + |\sigma|_2 - 2(\lambda+\mu) \quad \text{in } \Sigma_1''$$

The domains $\Sigma_2, \Sigma_2', \Sigma_2'', \Sigma_3, \Sigma_3', \Sigma_3''$ and the values of the function U_1^* at these domains are obtained from (5.2) and (5.3) by cyclic permutation of the subscripts.

The form of the functions U_I^* can be simplified substantially if a neighborhood of the unstressed state of radius r_0 is considered, where $r_0 = E / \sqrt{1+2\nu^2}$ (it is easy to see that the point (0, 0, 0) in the space of the variables $(|\sigma|_1, |\sigma|_2, |\sigma|_3)$ belongs to Σ, and r_0 is the distance from the point (0, 0, 0) to each of the planes bounding Σ).

Then in the neighborhood under consideration

$$\bar{U}^*(\sigma_i^p) = {}^1/_4\,\mu\left[|\sigma|_1^2 + |\sigma|_2^2 + |\sigma|_3^2 - \frac{\nu}{1+\nu}(|\sigma|_1 + |\sigma|_2 + |\sigma|_3)^2\right] + |\sigma|_3 + |\sigma|_2 + |\sigma|_3$$

for $\det \| \sigma_i^p \| \geqslant 0$.

If $\det \| \sigma_i^p \| < 0$

$$\bar{U}^*(\sigma_i^p) = {}^1/_4\,\mu\left[|\sigma|_1^2 + |\sigma|_2^2 + |\sigma|_3^2 - \frac{\nu}{1+\nu}(|\sigma|_1 + |\sigma|_2 - |\sigma|_3)^2\right] + |\sigma|_1 + |\sigma|_2 - |\sigma|_3$$

where the eigenvalues of the tensor $|\sigma|^{pq}$ are enumerated in decreasing order.

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